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Ising model on a checkerboard lattice in a magnetic field $H = \frac{1}{2}i\pi kT$

K Y Lin

Physics Department, National Tsing Hua University, Hsinchu, Taiwan 30043

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Abstract. The Lee-Yang formulae for the free energy and the magnetisation of the Ising model on a square lattice in an imaginary magnetic field $H = \frac{1}{2}i\pi kT$ are generalised to a checkerboard lattice.

1. Introduction

The free energy and the magnetisation of the Ising model on a square lattice in the presence of an imaginary magnetic field $H = \frac{1}{2}i\pi kT$ were first obtained by Lee and Yang (1952), although they never published their derivation. Their result has been rederived from several different approaches (Baxter 1966, McCoy and Wu 1967, Marshall 1971, Merlini 1974, Gaaff 1974, Wu 1986). McCoy and Wu (1967) generalised the result of Lee and Yang to a rectangular lattice, where the coupling constants along horizontal and vertical directions are different. Recently, Wu (1986) generalised the result of McCoy and Wu to a checkerboard lattice with crossing and four-spin interactions. The aim of the present paper is to generalise the well known Lee-Yang formulae to a checkerboard lattice with five different coupling constants, as shown in figure 1. Our lattice is different from the one considered by Wu such that the four coupling constants (J_1, J_2, J_3, J_4) along four sides of each square are different, while in his lattice $J_1 = J_3$, $J_2 = J_4$.

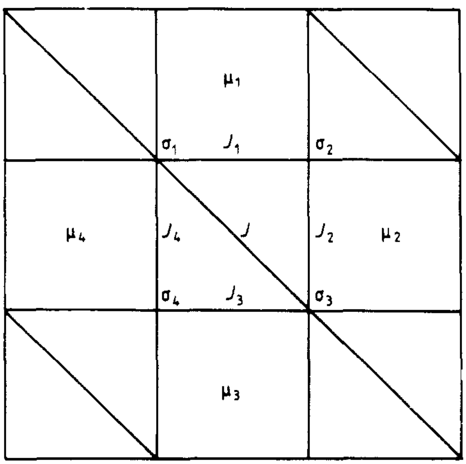

Figure 1. The checkerboard Ising lattice.

2. Equivalence with the eight-vertex model

Consider an Ising model of N spins σ_i on a checkerboard lattice with five coupling constants (see figure 1). Each spin carries one unit of magnetic moment. It has been pointed out by Wu (1986) that the Ising model on a general checkerboard lattice is equivalent to an eight-vertex model. Following Wu (1986) and Burkhardt (1979), we place the $\frac{1}{2}N$ dual spins μ_i in the squares without diagonal interaction. It is shown by Wu that the partition function of the Ising model Z is the same as that of the eight-vertex model Z^* in the dual space such that

$$Z = \sum_{\sigma_i=\pm 1} \prod B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{1}$$

where the product is taken over all squares with diagonal interaction, and

$$Z^* = \sum_{\mu_i=\pm 1} \prod W(\mu_1, \mu_2, \mu_3, \mu_4) \tag{2}$$

where the product is taken over all squares in the dual space (see figure 1).

The Boltzmann factors B are related to the vertex weights W by

$$2W_i = \sum_j X_{ij} B_j \tag{3}$$

where

$$\begin{aligned}
W_1 &= W(++++) & B_1 &= B(++++) \\
W_2 &= W(-+-+) & B_2 &= B(-+-+) \\
W_3 &= W(--++) & B_3 &= B(--++) \\
W_4 &= W(+--+) & B_4 &= B(+--+) \\
W_5 &= W(---+) & B_5 &= B(-+--) \\
W_6 &= W(-+--) & B_6 &= B(---+) \\
W_7 &= W(+---) & B_7 &= B(+---) \\
W_8 &= W(--+-) & B_8 &= B(--+-)
\end{aligned} \tag{4}$$

and X_{ij} are the elements of the following symmetric matrix:

$$\begin{bmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & + & - & - & - & - \\
+ & + & - & - & - & - & + & + \\
+ & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + \\
+ & - & - & + & - & + & + & - \\
+ & - & + & - & - & + & - & + \\
+ & - & + & - & + & - & + & -
\end{bmatrix}. \tag{5}$$

Here + (−) denotes +1 (−1). A special case of Wu's result was derived earlier by Giacomini (1985) using a different approach.

In the case of $H = \frac{1}{2}\mathrm{i}\pi kT$, we have

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1\sigma_2 \exp(-E/kT) \tag{6}$$

where

$$E = -J_1\sigma_1\sigma_2 - J_2\sigma_2\sigma_3 - J_3\sigma_3\sigma_4 - J_4\sigma_4\sigma_1 - J\sigma_1\sigma_3.$$

We define

$$\begin{aligned}
&x_i = \exp(-2J_i/kT) && y = \exp(-2J/kT) \\
&a = 1 - x_1x_2x_3x_4 && a' = 1 + x_1x_2x_3x_4 \\
&b = (x_2x_4 - x_1x_3)y && b' = (x_2x_4 + x_1x_3)y \\
&c = x_3x_4 - x_1x_2 && c' = x_3x_4 + x_1x_2 \\
&d = (x_2x_3 - x_1x_4)y && d' = (x_2x_3 + x_1x_4)y \\
&G = \tfrac{1}{2}(x_1x_2x_3x_4y)^{-1/2}.
\end{aligned} \tag{7}$$

It is straightforward to show that

$$\begin{aligned}
&W_1 = (a+b+c+d)G && W_2 = (a+b-c-d)G \\
&W_3 = (a-b-c+d)G && W_4 = (a-b+c-d)G \\
&W_5 = (a'-b'-c'+d')G && W_6 = (a'-b'+c'-d')G \\
&W_7 = (a'+b'+c'+d')G && W_8 = (a'+b'-c'-d')G.
\end{aligned} \tag{8}$$

It is simple to check that the free-fermion condition (Fan and Wu 1970)

$$W_1W_2 + W_3W_4 = W_5W_6 + W_7W_8 \tag{9}$$

is satisfied.

Similarly, in the case of $H = 0$, the Boltzmann factor is

$$B' = \exp(-E/kT) \tag{10}$$

and the corresponding eight-vertex weights are

$$\begin{aligned}
&W_1' = W_7 && W_2' = W_8 && W_3' = W_5 && W_4' = W_6 \\
&W_5' = W_3 && W_6' = W_4 && W_7' = W_1 && W_8' = W_2.
\end{aligned} \tag{11}$$

The free-fermion condition is again satisfied.

3. The free energy and the magnetisation

The free energy for an eight-vertex model has been calculated exactly by Fan and Wu (1970) when the free-fermion condition is satisfied. It follows from the equivalence with an eight-vertex model that the free energy of the Ising model in a magnetic field $H = \frac{1}{2}\mathrm{i}\pi kT$ is given by

$$\begin{aligned}
f = -H - (kT/16\pi^2)\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\theta\,\mathrm{d}\phi\,\log[&A + 2B\cos\theta + 2C\cos\phi + 2D \\
&\times\cos(\theta-\phi) + 2E\cos(\theta+\phi)]
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
A &= W_1^2 + W_2^2 + W_3^2 + W_4^2 = 4G^2(a^2+b^2+c^2+d^2) \\
B &= W_1 W_3 - W_2 W_4 = 4G^2(ad - bc). \\
C &= W_1 W_4 - W_2 W_3 = 4G^2(ac - bd) \\
D &= W_3 W_4 - W_7 W_8 = -4G^2(1-x_1^2)(1-x_3^2)x_2 x_4 y \\
E &= W_3 W_4 - W_5 W_6 = 4G^2(1-x_2^2)(1-x_4^2)x_1 x_3 y.
\end{aligned} \tag{13}$$

For a square lattice, we have $x_i = x$, $y = 1$ and our result reduces to that of Lee and Yang. For a rectangular lattice, we have $x_1 = x_3$, $x_2 = x_4$, $y = 1$ and our result reduces to that of McCoy and Wu (1967).

It is shown by Wu (1986) that the two-spin correlation function $\langle\sigma(0,0)\sigma(n,n)\rangle$, where $\sigma(i,j)$ is the spin located at the point (i,j) in figure 1, is given by

$$\langle\sigma(0,0)\sigma(n,n)\rangle = Z^{*n}/Z^* \tag{14}$$

where $Z^{*(n)}$ is the partition function of the eight-vertex model with vertex weights along a single row of n sites modified to new values. These new weights are obtained from (3) with the replacement.

$$B(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \to \sigma_1\sigma_3 B(\sigma_1,\sigma_2,\sigma_3,\sigma_4). \tag{15}$$

Consequently the magnetisation $I(W_i)$ of the Ising model on a checkerboard lattice at $H = \frac{1}{2}\mathrm{i}\pi kT$ is related to the spontaneous magnetisation $M(W_i')$ at $H = 0$ by

$$I(W_i) = M(W_i). \tag{16}$$

Syozi and Naya (1960) made a conjecture for the spontaneous magnetisation of the Ising model on a generalised square lattice which is a special case ($J = 0$) of our lattice. Their conjecture was confirmed recently (Lin and Fang 1985, Baxter 1986) and the result can be written in the form

$$M_0 = (N/D)^{1/8} \tag{17}$$

where

$$\begin{aligned}
N &= (W_1' + W_2' + W_3' - W_4')(W_1' + W_2' - W_3' + W_4')(W_1' - W_2' + W_3' + W_4') \\
&\qquad \times (-W_1' + W_2' + W_3' + W_4') \\
D &= 16 W_5' W_6' W_7' W_8'.
\end{aligned}$$

The magnetisation I_0 for $J = 0$ is obtained from (17) by the replacement $W_i' \to W_i$. The result is

$$I_0 = (N/D)^{1/8} \tag{18}$$

where

$$\begin{aligned}
N &= (a+b-c+d)(a+b+c-d)(a-b+c+d)(a-b-c-d) \\
D &= (a'+b'+c'+d')(a'+b'-c'-d')(a'-b'+c'-d')(a'-b'-c'+d').
\end{aligned}$$

In the special case of the square lattice ($x_i = x$, $y = 1$) we have

$$\begin{aligned}
&a = 1 - x^4 \qquad b = c = d = 0 \\
&a' = 1 + x^4 \qquad b' = c' = d' = 2x^2 \\
&I_0^8 = (1+x^2)^4/(1-x^2)^2(1+6x^2+x^4)
\end{aligned} \tag{19}$$

which is the well known Lee-Yang formula. In the special case of the rectangular lattice ($x_1 = x_3 = x$, $x_2 = x_4 = x'$, $y = 1$) we have

$$\begin{aligned} a &= 1-(xx')^2 & a' &= 1+(xx')^2 \\ b &= x'^2 - x^2 & b' &= x'^2 + x^2 \\ c &= d = 0 & c' &= d' = 2xx' \\ I_0^8 &= [(1+x^2)(1+x'^2)]^2/[(1+xx')^2+(x+x')^2][(1-xx')^2+(x-x')^2] \end{aligned} \tag{20}$$

which agrees with the result of McCoy and Wu (1967).

When $J \neq 0$, the spontaneous magnetisation is still unknown.

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